UDC 531.36

STABILITY OF STATIONARY ROTATION OF A HEAVY SOLID BODY WITH TWO ELASTIC RODS

PMM Vol. 40, № 1, 1976, pp. 55-64
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(Received August 25, 1975)

The example of the stability problem for stationary vertical rotation of a heavy "tuning fork" with a single point is used for showing that the method of stability investigation of a mechanical system stationary motions, based on the solution of the problem of minimum transformed potential energy of a system [1 — 6], is equally suitable for analyzing "trivial" and "nontrivial" stationary motions for which deformable elements of the system are, respectively, in the undeformed and the deformed state. Two methods for solving this problem are presented. Sufficient stability conditions that impose certain restrictions from below on the stiffness of rods and also from above and below on the angular velocity of the system uniform rotation are obtained and analyzed.

The general statement of the problem of motion and stability of an elastic body with a cavity containing a fluid was presented by Rumiantsev [1]. The theorem on stability proved here is an extension of the Routh theorem to systems with distributed parameters, and reduces the problem of stability of stationary motion to that of minimum (transformed) potential energy W of the system. Solutions of a number of problems on the stability of stationary motions of a solid body with elastic rods and fluid in potential force fields appear in [2 — 5]. Two methods for establishing conditions for positive definiteness of the second variation $\delta^2 W$ in investigations of motion stability of mechanical systems consisting of absolutely rigid bodies and material points with attached deformable elastic and fluid bodies are presented and illustrated in [6] on the example of solutions of specific mechanical problems.

The problem of stability of stationary motions of mechanical systems with distributed parameters were investigated in [7 — 10] (see the bibliography in [7 — 10]). The problem of stability of the "nontrivial" state of relative equilibrium of an absolutely rigid body with elastic rods is considered in [9], where the "trivial" and "nontrivial" equilibrium states of the system are characterized by the undeformed and deformed states, respectively. Only trivial states of relative equilibrium of rigid bodies with elastic rods were investigated in [2 — 6] and [7, 8]. It is noted in [9] that the previously used method based on the direct Liapunov method cannot be applied for solving problems of stability of the nontrivial state of relative equilibrium of the system, and another method of solution, based on the expansion of elastic displacements of rods in series of some complete system of functions is proposed. A finite number of first terms was retained in such series without any mathematical substantiation, and the system with distributed parameters is essentially reduced to some system with a finite number of degrees of freedom.

1. Let us consider the motion of a rigid body with one fixed point and two attached thin rectilinear elastic rods, moving in a uniform gravity field.

We introduce two rectangular coordinate systems: one inertial $Oxyz$ with its origin at the fixed point O of the body and the z-axis directed vertically upward, the other $Ox_1x_2x_3$ which is moving with its axes coinciding with the principal axes of the rigid body ellipsoid of inertia for point O. Let $\mathbf{i}_\nu$ $(\nu = 1, 2, 3)$ be unit vectors of axes x_ν, and γ the unit vector of the z-axis whose projections on the x_ν-axes are γ_ν.

We assume that two identical rods of length l are attached to the body at points N_1 and N_2 defined by coordinates N_1 $(0, a, b)$ and N_2 $(0, —a, b)$. In the undeformed state the two rods lie in the plane $x_1=0$ with their free ends pointing in the same direction parallel to the x_3-axis. The planes passing through the geometric axes of rods and parallel to planes $x_1 = 0$ and $x_2 = 0$ are the planes of symmetry of the rods.

We denote by

$$\boldsymbol{u}_j(t, s) = u_{j1}(t, s)\,\boldsymbol{i}_1 + u_{j2}(t, s)\,\boldsymbol{i}_2 + u_{j3}(t, s)\,\boldsymbol{i}_3$$
$$0 \leqslant s \leqslant l \quad t \geqslant t_0 \quad (j = 1, 2)$$

the vectors of elastic displacements of points of the rod axes. The condition of rod inextensibility is expressed by formulas [11]

$$u_{j3}' = -\tfrac{1}{2}(u_{j1}'^2 + u_{j2}'^2), \quad j = 1, 2 \quad (u' = \partial u / \partial s) \tag{1.1}$$

and the condition for the rods to be fixed at one end to the body provides the boundary conditions

$$u_{j1} = u_{j2} = 0, \quad u_{j1}' = u_{j2}' = \quad \text{for} \quad s = 0,\ t \geqslant t_0 \tag{1.2}$$

It follows from (1.1) that u_{13} and u_{23} are quantities of the second order of smallness, if u_{j1}, u_{j2}, u_{j1}' and u_{j2}' $(j = 1, 2)$ are taken as quantities of the first order. Note that equalities (1.1) represent the condition of rod inextensibility that is accurate only to terms of second order of smallness with respect to the indicated magnitudes.

We define the potential energy of elastic deformation by formula [11]

$$\Pi_d = \frac{1}{2}\int_0^l [EI_2(u_{11}''^2 + u_{21}''^2) + EI_1(u_{12}''^2 + u_{22}''^2)]\,ds \tag{1.3}$$

where E is the Young modulus; I_1 and I_2 are the moments of inertia of the rod profile about straight lines drawn through its center of gravity parallel to the x_1- and x_2-axes, respectively, and EI_1 and EI_2 are flexural rigidities.

The potential energy of the force of gravity is

$$\Pi_g = Mg(x_{10}\gamma_1 + x_{20}\gamma_2 + x_{30}\gamma_3) + \tag{1.4}$$
$$g\sigma\rho\int_0^l [\gamma_1(u_{11} + u_{21}) + \gamma_2(u_{12} + u_{22}) - \tfrac{1}{2}\gamma_3(l - s)(u_{11}'^2 + u_{12}'^2 + u_{21}'^2 + u_{22}'^2)]\,ds$$

where M is the mass of the complete system; x_{10}, x_{20} and x_{30} are the coordinates of the system center of mass in the undeformed state; g is the acceleration of gravity; σ is the area of the rod cross section, and ρ is the rod density. Note that formulas (1.1) and (1.2) were used in the derivation of formula (1.4).

The considered system admits integrals of energy $T + \Pi = \text{const}$ and of areas $\mathbf{G}\cdot\gamma = k = \text{const}$, where T and $\Pi = \Pi_d + \Pi_g$ are the kinetic and the potential energies of the system, and $\mathbf{G}$ is the vector of kinetic moment of the system about point O.

We introduce into the analysis a rectangular system of coordinates $Ozz'z''$ that rotates at some angular velocity Ω about the z-axis. Denoting by G_r the vector of kinetic moment of the system about point O in its motion relative to axes $Ozz'z''$, we represent the integral of areas in the form $G_r \cdot \gamma + J\Omega = k$, where J is the moment of inertia of the system about the z-axis defined by

$$J = J_1\gamma_1^2 + J_2\gamma_2^2 + J_3\gamma_3^2 + \sigma\rho \int_0^l \{(u_{11}^2 + u_{21}^2)(1-\gamma_1^2) + (u_{12}^2 + u_{22}^2)(1-\gamma_2^2) - (l-s)[b + {}^1/_2(l+s)](u_{11}'^2 + u_{12}'^2 + u_{21}'^2 + u_{22}'^2)(1-\gamma_3^2) - 2(u_{11}u_{12} + u_{21}u_{22})\gamma_1\gamma_2 + a(l-s)(u_{11}'^2 + u_{12}'^2 - u_{21}'^2 - u_{22}'^2)\gamma_2\gamma_3 - 2(b+s)[(u_{11}+u_{21})\gamma_1 + (u_{12}+u_{22})\gamma_2]\gamma_3 + 2a[(u_{12}-u_{22})(1-\gamma_2^2) + (u_{21}-u_{11})\gamma_1\gamma_2)\} ds \tag{1.5}$$

and J_1, J_2 and J_3 are the principal moments of inertia of the undeformed system relative to axes x_1, x_2 and x_3

We select Ω so that at any instant of time the equality $G_r \cdot \gamma = 0$ be satisfied. We then have $J\Omega = k$, and the energy integral may be presented in the form $T_r + W =$ const, where T_r is the kinetic energy of the system relative motion and W is the transformed potential energy of the system defined by

$$W = \frac{k^2}{2J} + \Pi \tag{1.6}$$

In what follows instead of W we consider the functional $W_* = W + {}^1/_2\lambda \cdot (\gamma^2 - 1)$, where λ is the indeterminate Lagrange multiplier. From (1.3),(1.4) and (1.6) we obtain for W_* the expression

$$W_* = \frac{k^2}{2J} + Mg(x_{10}\gamma_1 + x_{20}\gamma_2 + x_{30}\gamma_3) + {}^1/_2\lambda(\gamma_1^2+\gamma_2^2+\gamma_3^2-1) + \sigma\rho\int_0^l \{g[\gamma_1(u_{11}+u_{21}) + \gamma_2(u_{12}+u_{22}) - {}^1/_2\gamma_3(l-s)(u_{11}'^2 + u_{12}'^2 + u_{21}'^2 + u_{22}'^2)] + {}^1/_2[E_*I_2(u_{11}''^2 + u_{21}''^2) + E_*I_1(u_{12}''^2 + u_{22}''^2)]\} ds \tag{1.7}$$

$$(E = \sigma\rho E_*)$$

2. We obtain the equations of stationary motions of the system and reasonable boundary conditions by computing and equating to zero the first variation δW_*. These equations are of the form

$$Mgx_{10} - \Omega^2(J_1 - \lambda_*)\gamma_1 + \sigma\rho\int_0^l \{[g + \Omega^2(b+s)\gamma_3](u_{11}+u_{21}) + \Omega^2[(u_{11}^2+u_{21}^2)\gamma_1 + (u_{11}u_{12}+u_{21}u_{22})\gamma_2 + a(u_{11}-u_{21})\gamma_2]\} ds = 0 \tag{2.1}$$

$$Mgx_{20} - \Omega^2(J_2 - \lambda_*)\gamma_2 + \sigma\rho\int_0^l \{[g + \Omega^2(b+s)\gamma_3](u_{12}+u_{22}) + \Omega^2[(u_{11}u_{12}+u_{21}u_{22})\gamma_1 + a(u_{11}-u_{21})\gamma_1 + (u_{12}^2+u_{22}^2)\gamma_2 + 2a(u_{12}-u_{22})\gamma_2 + \frac{1}{2}a(l-s)(u_{21}'^2 + u_{22}'^2 - u_{11}'^2 - u_{12}'^2)\gamma_3]\} ds = 0$$

$$Mgx_{30} - \Omega^2 (J_3 - \lambda_*) \gamma_3 + \sigma\rho \int_0^l \Big\{ -\frac{1}{2}(l-s)[g + \Omega^2(2b+l+s)\gamma_3] \times (u_{11}'^2 + u_{12}'^2 + u_{21}'^2 + u_{22}'^2) + \Omega^2 [(b+s)(u_{11}+u_{21})\gamma_1 + (b+s)(u_{12}+u_{22})\gamma_2 + \frac{1}{2} a (l-s)(u_{21}'^2 + u_{22}'^2 - u_{11}'^2 - u_{12}'^2)\gamma_2] \Big\} \times ds = 0$$

$$E_* I_2 u_{11}^{\mathrm{IV}} + \{[a\Omega^2\gamma_2\gamma_3 + g\gamma_3 - {}^1/_2\Omega^2 (2b + l + s)(1 - \gamma_3^2)] \times (l - s) u_{11}'\}' + \Omega^2 [u_{12}\gamma_1\gamma_2 - u_{11}(1 - \gamma_1^2)] + \{g + \Omega^2 [a\gamma_2 + (b+s)\gamma_3]\}\gamma_1 = 0$$

$$E_* I_1 u_{12}^{\mathrm{IV}} + \{[a\Omega^2\gamma_2\gamma_3 + g\gamma_3 - {}^1/_2\Omega^2 (2b + l + s)(1 - \gamma_3^2)] \times (l - s) u_{12}'\}' + \Omega^2 [u_{11}\gamma_1\gamma_2 - u_{12}(1 - \gamma_2^2)] + g\gamma_2 + \Omega^2 [(b + s)\gamma_2\gamma_3 - a(1 - \gamma_2^2)] = 0$$

$$E_* I_2 u_{21}^{\mathrm{IV}} - \{[a\Omega^2\gamma_2\gamma_3 - g\gamma_3 + {}^1/_2\Omega^2 (2b + l + s)(1 - \gamma_3^2)] \times (l - s) u_{21}'\}' + \Omega^2 [u_{22}\gamma_1\gamma_2 - u_{21}(1 - \gamma_1^2)] + \{g + \Omega^2 [(b + s)\gamma_3 - a\gamma_2]\}\gamma_1 = 0$$

$$E_* I_1 u_{22}^{\mathrm{IV}} - \{[a\Omega^2\gamma_2\gamma_3 - g\gamma_3 + {}^1/_2\Omega^2 (2b + l + s)(1 - \gamma_3^2)] \times (l - s) u_{22}'\}' + \Omega^2 [u_{21}\gamma_1\gamma_2 - u_{22}(1 - \gamma_2^2)] + g\gamma_2 + \Omega^2 [(b + s)\gamma_2\gamma_3 + a(1 - \gamma_2^2)] = 0$$

$$(\Omega = kJ^{-1}, \ \lambda_* = \lambda\Omega^{-2})$$

$$u_{11}'' = u_{12}'' = u_{21}'' = u_{22}'' = 0, \quad u_{11}''' = u_{12}''' = u_{21}''' = u_{22}''' = 0 \quad \text{for } s = l \qquad (2.2)$$

Boundary conditions (2.2) must be supplemented by conditions (1.1).

For $x_{10} = x_{20} = 0$ Eqs. (2.1) and boundary conditions (1.1) and (2.2) admit the following particular solution:

$$\gamma_1^\circ = \gamma_2^\circ = 0, \ \gamma_3^\circ = 1, \quad u_{11}^\circ = u_{21}^\circ \equiv 0, \ u_{12}^\circ = -u_{22}^\circ = u_0(s) \qquad (2.3)$$

where $u = u_0(s)$ is the solution of the boundary value problem

$$E_* I_1 u_0^{\mathrm{IV}} + g[(l - s) u_0']' - \Omega_0^2 (u_0 + a) = 0 \qquad (2.4)$$

$$u_0(0) = u_0'(0) = u_0''(l) = u_0'''(l) = 0$$

with

$$\lambda = \lambda_* \Omega^2{}_0 = J_3\Omega_0^2 - Mgx_{30} + \sigma\rho \int_0^l (l - s)[g + \Omega_0^2 (2b + l + s)] u_0'^2 ds \qquad (2.5)$$

Solution (2.3) defines the rotation of the system at constant angular velocity $\Omega_0 = k_0 J_0^{-1}$ about a vertical axis that coincides with the x_3-axis of the ellipsoid of inertia of the rigid body for its fixed point. In the above formula k_0 and J_0 are constants of the integrals of areas and of the system moments of inertia about the z-axis for the stationary motion (2.3).

3. Let us investigate the stability (definition of stability appears in Sect. 4 of [2]) of motion (2.3) on the assumption that it is unperturbed.

The conditions of stability are obtained from the theorem in [1], as the conditions of positive definiteness of the second variation $\delta^2 W_*$ for solution (2.3) in metric with respect to which the functional W_* is continuous [2]. We shall consider two methods [6] for establishing the positive definiteness of $\delta^2 W_*$.

For the perturbed motion we set $\gamma_3 = 1 + \delta\gamma_3$, $u_{12} = u_0(s) + w_{12}$ and $u_{22} = -u_0(s) + w_{22}$, and retain previous notation for the remaining quantities. The equality $\gamma^2 = 1$ implies that $\delta\gamma_3 = 0$ with an accuracy to terms of order of smallness higher than the first. Hence it is possible to assume in the calculation of $\delta^2 W_*$ that $\gamma_3 = 1$.

From (1.7), (1.5) and (2.3) — (2.5) we obtain

$$\delta^2 W_* = \Omega_0^2 [(\lambda_* - J_1)\gamma_1^2 + (\lambda_* - J_2)\gamma_2^2] + \sigma\rho \int_0^l \{E_* I_2 (u_{11}''^2 + u_{21}''^2) + E_* I_1 (w_{12}''^2 + w_{22}''^2) - g(l-s)(u_{11}'^2 + u_{21}'^2 + w_{12}'^2 + w_{22}'^2) - \Omega_0^2 (u_{11}^2 + u_{21}^2 + w_{12}^2 + w_{22}^2) + 2[g + \Omega_0^2 (b+s)][(u_{11} + u_{21})\gamma_1 + (w_{12} + w_{22})\gamma_2] - 2\Omega_0^2 [a(l-s) u_0' (w_{12}' + w_{22}') - (2a + u_0) u_0 \gamma_2] \gamma_2\} ds + 4J_0^{-1}\Omega_0^2 \left\{\sigma\rho \int_0^l (a + u_0)(w_{12} - w_{22}) ds\right\}^2 \tag{3.1}$$

We rewrite (3.1) in the form

$$\delta^2 W_* = U(\gamma_1, \gamma_2, u_{11}, u_{21}, w_{12}, w_{22}) + U_1(u_{11}, u_{21}) + U_2(w_{12}, w_{22}) \tag{3.2}$$

$$U = 4J_0^{-1}\Omega_0^2 \left\{\sigma\rho \int_0^l (a + u_0)(w_{12} - w_{22}) ds\right\}^2 + (\lambda_* - J_1)\Omega_0^2 \left\{\gamma_1 + (\lambda_* - J_1)^{-1}\Omega_0^{-2}\sigma\rho \int_0^l [g + \Omega_0^2(b+s)](u_{11} + u_{21}) ds\right\}^2 + (\lambda_* - J_2 + A)\Omega_0^2 \times \left\{\gamma_2 + (\lambda_* - J_2 + A)^{-1}\Omega_0^{-2}\sigma\rho \int_0^l \{[g + \Omega_0^2(b+s)](\omega_{12} + w_{22}) - a\Omega_0^2 (l-s) u_0' (w_{12}' + w_{22}')\} ds\right\}^2 \tag{3.3}$$

$$U_1 = \sigma\rho \int_0^l \{E_* I_2 (u_{11}''^2 + u_{21}''^2) - g(l-s)(u_{11}'^2 + u_{21}'^2) - \Omega_0^2 (u_{11}^2 + u_{21}^2)\} ds - (\lambda_* - J_1)^{-1}\Omega_0^{-2} \left\{\sigma\rho \int_0^l [g + \Omega_0^2 (b+s)](u_{11} + u_{21}) ds\right\}^2 \tag{3.4}$$

$$U_2 = \sigma\rho \int_0^l \{E_* I_1 (w_{12}''^2 + w_{22}''^2) - g(l-s)(w_{12}'^2 + w_{22}'^2) - \Omega_0^2 (w_{12}^2 + w_{22}^2)\} ds - (\lambda_* - J_2 + A)^{-1}\Omega_0^{-2} \times \left\{\sigma\rho \int_0^l \{g + \Omega_0^2 (b+s) + a\Omega_0^2 [(l-s) u_0']'\} (w_{12} + w_{22}) ds\right\}^2 \tag{3.5}$$

$$A = 2\sigma\rho \int_0^l (2a + u_0) u_0 ds \tag{3.6}$$

If conditions $\lambda_*-J_1>0$ and $\lambda_*-J_2+A>0$ that represent the sufficient conditions of stability of uniform vertical rotation (2.3) of a heavy rigid body with two identical undeformable rods bent according to the law defined by (2.3) and (2.4), are satisfied, then using the Cauchy-Buniakowski inequality, from (3.4) and (3.5) we obtain inequalities

$$U_1 \geqslant \sigma\rho\int_0^l \{E_* I_2 (u_{11}''^2 + u_{21}''^2) - g(l-s)(u_{11}'^2 + u_{21}'^2) - \Omega_0^2 (u_{11}^2 + u_{21}^2) - (\lambda_* - J_1)^{-1}\Omega_0^{-2} h_1 (u_{11} + u_{21})^2\}\, ds \tag{3.7}$$

$$U_2 \geqslant \sigma\rho\int_0^l \{E_* I_1 (w_{12}''^2 + w_{22}''^2) - g(l-s)(w_{12}'^2 + w_{22}'^2) - \Omega_0^2 (w_{12}^2 + w_{22}^2) - (\lambda_* - J_2 + A)^{-1}\Omega_0^{-2} h_2 (w_{12} + w_{22})^2\}\, ds \tag{3.8}$$

$$h_1 = \sigma\rho\int_0^l [g + \Omega_0^2 (b+s)]^2\, ds, \qquad h_2 = \sigma\rho\int_0^l \{g + \Omega_0^2(b+s) + a\Omega_0^2 [(l-s)u_0']'\}^2\, ds \tag{3.9}$$

Let us consider now the following variational problem. Find the minimum $\varkappa l^{-4}$ of functional

$$F(u) = \int_0^l u''^2 ds \left\{\int_0^l (u^2 + \sigma u'^2)\, ds\right\}^{-1} \tag{3.10}$$

in the class of functions $u(s)$ ($0 \leqslant s \leqslant l$) that are continuously differentiable up to and including the fourth order and satisfy the conditions $u(0) = u'(0) = 0$.

Setting now $s = lx$ for the determination of the constant $\varkappa$ we obtain the problem of eigenvalues

$$\frac{d^4 u}{dx^4} + \varkappa\left(\nu \frac{d^2 u}{dx^2} - u\right) = 0 \qquad (\nu = \sigma l^{-2})$$

$$u(0) = \left(\frac{du}{dx}\right)_{x=0} = \left(\frac{d^2 u}{dx^2}\right)_{x=1} = 0, \quad \left(\frac{d^3 u}{dx^3}\right)_{x=1} + \nu\varkappa\left(\frac{du}{dx}\right)_{x=1} = 0$$

whose characteristic equation is of the form

$$\Delta(\varkappa) = \alpha^2 + \beta^2 + \nu\varkappa(\beta^2 - \alpha^2) + [(1+\alpha^2)\beta^2 + \nu\varkappa(1-\beta^2)] \times \cos\alpha \operatorname{ch}\beta - \alpha\beta(\beta^2 - \alpha^2 + 2\nu\varkappa)\sin\alpha \operatorname{sh}\beta = 0 \tag{3.11}$$

$$2\alpha = \nu\varkappa + \sqrt{4\varkappa + \nu^2\varkappa^2}, \quad 2\beta = -\nu\varkappa + \sqrt{4\varkappa + \nu^2\varkappa^2}$$

The sought $\varkappa$ is equal to the smallest (positive) root of Eq. (3.11).

From (3.10) we obtain the inequality

$$\int_0^l u''^2 ds \geqslant \varkappa l^{-4} \int_0^l (u^2 + \sigma u'^2)\, ds \tag{3.12}$$

and, taking it into account, from (3.7) and (3.8) we obtain inequalities

$$U_1 \geqslant \sigma\rho\int_0^l \{[E_* I_2 l^{-4}\sigma\varkappa - g(l-s)](u_{11}'^2 + u_{21}'^2) + [E_* I_2 l^{-4}\varkappa - \Omega_0^2 - (\lambda_* - J_1)^{-1}\Omega_0^{-2} h_1](u_{11}^2 + u_{21}^2) - 2(\lambda_* - J_1)^{-1}\Omega_0^{-2} h_1 u_{11} u_{21}\}\, ds$$

$$U_2 \geqslant \sigma\rho \int_0^l \{[E_* I_1 l^{-4}\sigma\varkappa - g(l-s)](w_{12}'^2 + w_{22}'^2) +$$
$$[E_* I_1 l^{-4}\varkappa - \Omega_0^2 - (\lambda_* - J_2 + A)^{-1}\Omega_0^{-2} h_2](w_{12}^2 + w_{22}^2) -$$
$$2(\lambda_* - J_2 + A)^{-1}\Omega_0^{-2} h_2 w_{12} w_{22}\}\, ds$$

This, with allowance for (3. 2) — (3. 5) implies that inequalities

$$E_* I_2 l^{-4} \sigma\varkappa > gl, \quad E_* I_1 l^{-4} \sigma\varkappa > gl \tag{3. 13}$$

$$\lambda_* - J_1 > 2h_1\Omega_0^{-2}(E_* I_2 l^{-4}\varkappa - \Omega_0^2)^{-1} > 0 \tag{3. 14}$$

$$\lambda_* - J_2 + A > 2h_2\Omega_0^{-2}(E_* I_1 l^{-4}\varkappa - \Omega_0^2)^{-1} > 0$$

represent sufficient conditions for the positive definiteness of functional $\delta^2 W_*$. In accordance with the theorem in [1] we conclude that (3. 13) and (3. 14) are sufficient conditions of stability [2] of the unperturbed motion (2. 3).

Conditions (3. 13) impose certain restrictions from below on the rigidity of rods, while conditions (3. 14) impose on Ω_0^2 restrictions from below, as well as from above. This feature is characteristic of systems with distributed parameters [2, 4, 6]. Constants λ_*, A, h_1 and h_2 in (3. 13) and (3. 14) are computed by formulas (2. 5), (3. 6) and (3. 9) for a known solution $u = u_0(s)$ of problem (2. 4).

4. Let us describe another method [6] of solving the problem of minimum of functional $\delta^2 W_*$.

We shall consider formula (3. 1) as functional for fixed γ_1 and γ_2 which will be taken as parameters. Let us determine functions u_{11}, u_{21}, w_{12} and w_{22} for which the value of (3. 1) is stationary. The stationarity condition (4. 1) yields the boundary value problems

$$E_* I_2 u_{11}^{IV} + g[(l-s)u_{11}']' - \Omega_0^2 u_{11} + [g + \Omega_0^2(b+s)]\gamma_1 = 0 \tag{4. 1}$$
$$u_{11}(0) = u_{11}'(0) = u_{11}''(l) = u_{11}'''(l) = 0$$

$$E_* I_2 u_{21}^{IV} + g[(l-s)u_{21}']' - \Omega_0^2 u_{21} + [g + \Omega_0^2(b+s)]\gamma_1 = 0 \tag{4. 2}$$
$$u_{21}(0) = u_{21}'(0) = u_{21}''(l) = u_{21}'''(l) = 0$$

$$E_* I_1 w_{12}^{IV} + g[(l-s)w_{12}']' - \Omega_0^2 w_{12} + 4J_0^{-1}\Omega_0^2(a+u_0)\sigma\rho \times \tag{4. 3}$$
$$\int_0^l (a+u_0)(w_{12} - w_{22})\, ds + g\gamma_2 + \Omega_0^2\{b + s + a[(l-s)u_0']'\}\gamma_2 = 0$$

$$w_{12}(0) = w_{12}'(0) = w_{12}''(l) = w_{12}'''(l) = 0$$

$$E_* I_1 w_{22}^{IV} + g[(l-s)w_{22}']' - \Omega_0^2 w_{22} - 4J_0^{-1}\Omega_0^2(a+u_0)\sigma\rho \times \tag{4. 4}$$
$$\int_0^l (a+u_0)(w_{12} - w_{22})\, ds + g\gamma_2 + \Omega_0^2\{b + s + a[(l-s)u_0']'\}\gamma_2 = 0$$

$$w_{22}(0) = w_{22}'(0) = w_{22}''(l) = w_{22}'''(l) = 0$$

When conditions

$$E_* I_1 l^{-4}\varkappa > \Omega_0^2, \quad E_* I_1 l^{-4}\varkappa\sigma > gl \tag{4. 5}$$

are satisfied, the solutions of problems (4.3) and (4.4) are the same: $w_{12} \equiv w_{22}$.

In fact, by subtracting (4.4) from (4.3) we obtain for $w = w_{12} - w_{22}$ the homogeneous boundary value problem

$$E_* I_1 w^{IV} + g\,[(l-s)\,w']' - \Omega_0^2 w + 4J_0^{-1}\Omega_0^2 (a+u_0)\,\sigma\rho \int_0^l (a+u_0)\,w\,ds = 0 \qquad (4.6)$$

$$w\,(0) = w'\,(0) = w''\,(l) = w'''\,(l) = 0$$

Multiplying this equation termwise by w and integrating from 0 to l with respect to s, we obtain

$$\Phi(w) \equiv \int_0^l [E_* I_1 w''^2 - g\,(l-s)\,w'^2 - \Omega_0^2 w^2]\,ds + 4J_0^{-1}\Omega_0^2\sigma\rho \left\{\int_0^l (a+u_0)\,w\,ds\right\}^2 = 0$$

whose left-hand part is estimated on the basis of (3.12) as follows:

$$\Phi\,(w) \geqslant \int_0^l [(E_* I_1 l^{-4}\sigma\varkappa - gl)\,w'^2 + (E_* I_1 l^{-4}\varkappa - \Omega_0^2)\,w^2]\,ds$$

It follows from this that when conditions (4.5) are satisfied, problem (4.6) has the trivial solution $w \equiv 0$, hence $w_{12} \equiv w_{22}$.

We represent the solution of problems (4.1) — (4.4) in the form

$$u_{11} = u_{21} = \gamma_1 u_*, \qquad w_{12} = w_{22} = \gamma_2 w_* \qquad (4.7)$$

where u_* and w_* are solutions of the boundary value problems

$$E_* I_2 u_*^{IV} + g\,[(l-s) u_*']' - \Omega_0^2 u_* + g + \Omega_0^2\,(b+s) = 0 \qquad (4.8)$$

$$u_*\,(0) = u_*'\,(0) = u_*''\,(l) = u_*'''\,(l) = 0$$

$$E_* I_2 w_*^{IV} + g\,[(l-s)\,w_*']' - \Omega_0^2\,w_* + g + \Omega_0^2\,\{b + s + a\,[(l-s)\,u_0']'\} = 0 \qquad (4.9)$$

$$w_*(0) = w_*'\,(0) = w_*''\,(l) = w_*'''\,(l) = 0.$$

Setting in (3.1) $u_{11} = \gamma_1 u_* + v_{11}$, $u_{21} = \gamma_1 u_* + v_{21}$, $w_{12} = \gamma_2 w_* + v_{12}$ and $w_{22} = \gamma_2 w_* + v_{22}$, we obtain

$$\begin{aligned}\delta^2 W_* = {} & [(\lambda_* - J_1)\,\Omega_0^2 + P_1]\,\gamma_1^2 + [(\lambda_* - J_2 + A)\,\Omega_0^2 + P_2]\,\gamma_2^2 + \\ & \sigma\rho \int_0^l \{E_* I_2 (v_{11}''^2 + v_{21}''^2) + E_* I_1 (v_{12}''^2 + v_{22}''^2) - g\,(l-s)\,(v_{11}'^2 + v_{21}'^2 + \\ & v_{12}'^2 + v_{22}'^2) - \Omega_0^2\,(v_{11}^2 + v_{21}^2 + v_{12}^2 + v_{22}^2)\}\,ds + 4J_0^{-1}\Omega_0^2 \times \\ & \left\{\sigma\rho \int_0^l (a+u_0)\,(v_{12} - v_{22})\,ds\right\}^2\end{aligned} \qquad (4.10)$$

where

$$\begin{aligned}P_1 &= 2\sigma\rho \int_0^l [g + \Omega_0^2\,(b+s)]\,u_* ds \\ P_2 &= 2\sigma\rho \int_0^l \{g + \Omega_0^2\,(b + s + a\,[(l-s)\,u_0']')\}\,w_* ds\end{aligned} \qquad (4.11)$$

From (4.10) with allowance for (3.12) we obtain the inequality

$$\delta^2 W_* \geqslant [(\lambda_* - J_1)\,\Omega_0{}^2 + P_1]\,\gamma_1{}^2 + [(\lambda_* - J_2 + A)\,\Omega_0{}^2 + P_2]\,\gamma_2{}^2 +$$
$$4J_0^{-1}\Omega_0{}^2\left\{\sigma\rho\int_0^l (a+u_0)(v_{12}-v_{22})\,ds\right\}^2 +$$
$$\sigma\rho\int_0^l\{[E_* I_2 l^{-4}\sigma\varkappa - g(l-s)](v_{11}'^2 + v_{21}'^2) + [E_* I_1 l^{-4}\sigma\varkappa - g(l-s)]\times$$
$$(v_{12}'^2 + v_{22}'^2) + (E_* I_2 l^{-4}\varkappa - \Omega_0{}^2)(v_{11}^2 + v_{21}^2) +$$
$$(E_* I_1 l^{-1}\varkappa - \Omega_0{}^2)(v_{12}^2 + v_{22}^2)\}\,ds$$

which yields conditions of stability of motion (2. 3) in the form of the sufficient conditions of the positive definiteness of functional $\delta^2 W_*$

$$E_* I_2 l^{-4}\sigma\varkappa > gl, \quad E_* I_1 l^{-4}\sigma\varkappa > gl \tag{4.12}$$

$$E_* I_2 l^{-4}\varkappa > \Omega_0{}^2, \quad E_* I_1 l^{-4}\varkappa > \Omega_0{}^2 \tag{4.13}$$

$$(\lambda_* - J_1)\,\Omega_0{}^2 + P_1 > 0, \quad (\lambda_* - J_2 + A)\,\Omega_0{}^2 + P_2 > 0 \tag{4.14}$$

The mechanical meaning of conditions (4. 12) — (4. 14) is the same as that of (3. 13) and (3. 14), namely: conditions (4. 12) impose on the rigidity of rods restrictions from below, conditions (4. 13) impose on $\Omega_0{}^2$ restrictions from above, and conditions (4. 14) impose on the latter restrictions from below. To compute the constants P_1 and P_2 in (4. 14) by formulas (4. 11) it is necessary to know not only the solution of problem (2.4), but also the solutions of problems (4. 8) and (4. 9). For an unlimited increase of rod rigidity (for $E \to \infty$) we obtain at the limit from (3. 13) and (3. 14), as well as from (4.12)-(4. 14) the known sufficient conditions $\lambda_* - J_1 > 0$ and $\lambda_* - J_2 + A > 0$ of the stability of uniform vertical rotation in a homogeneous gravitation force field of an invariable system consisting of a rigid body with a single fixed point and two identical undeformable rods bent in accordance with the law defined by (2. 3) and (2. 4) and rigidly attached to it.

When the gravity force field is absent, conditions (3. 14) and (4. 14) can be derived in the explicit form, since then the solutions of the boundary problems (2. 4), (4. 8) and (4. 9) are expressed in terms of elementary functions.

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Translated by J. J. D.

UDC 531.36

ON TWO-DIMENSIONAL ELECTRO-GASDYNAMIC FLOWS WITH ALLOWANCE FOR THE INERTIA OF CHARGED PARTICLES

PMM Vol. 40, № 1, 1976, pp. 65-73
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(Received June 6, 1975)

Electro-gasdynamic flows in inertia-free approximation and those with allowance for inertia forces are investigated. Conditions under which inertia effects are considerable, are determined. Simple analytical solutions are derived for systems of electro-gasdynamic equations that describe the motion of particles in a uniform external electric field in the presence of tangential discontinuity of gasdynamic velocity at the half-plane boundary. The possibility of reverse current generation, i.e. of the return of particles to the emitter is demonstrated. Obtained results are compared with data related to inertia-free approximation. A numerical method is developed for solving the complete system of equations of electro-gasdynamics with allowance for particle inertia. The proposed method is used for investigating the expansion of electro-gasdynamic streams in channels. Results of numerical calculations for various values of controlling parameters are presented. Effects of inertia are set appart.

In many applications (such as electron-ion technology, electrically charged jet streams of aircraft engines) solid or fluid particles in a gasdynamic stream become electrically charged, and it is necessary to investigate two-phase electro-gasdynamic flows. General equations that define the electro-gasdynamic flow of a mixture of inert gas, particles, and ions appear in [1].

If the charged particle inertia is small, two-phase flows can be investigated by the method developed for solving equations of electro-gasdynamics with the Ohm law formulated in the inertia-free approximation [2 — 4]. Investigation of such two-dimensional